Propagation of waves of finite amplitude along a duct of non-uniform cross-section

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(Received 15 August 1959 and in revised form 16 February 1960)

This paper is concerned with the propagation of a simple wave along a duct of initially constant and then slowly varying cross-section. The equations of motion are linearized in the deviations from the simple wave solution. The solution for the propagation of a centred simple rarefaction wave is obtained in closed form and compared with the results of step-by-step calculations. The results are also found to be good when the area change is not small. Some remarks on boundary-layer influence are included.

1. Introduction

In this paper, the method of small parameters is applied to the system of partial differential equations which govern unsteady flow through ducts of variable cross-section (see Rudinger 1950)

$$\frac{\partial r}{\partial \alpha} \equiv \frac{\partial r}{\partial t} + (V+a)\frac{\partial r}{\partial x} = -\frac{a}{2}\frac{D\ln A}{Dt} + \frac{a}{2(k-1)c_p}\frac{\partial S}{\partial \alpha} + F_1(a, V, x, t) \tag{1}$$

$$\frac{\partial s}{\partial \beta} \equiv \frac{\partial s}{\partial t} + (V-a)\frac{\partial s}{\partial x} = -\frac{a}{2}\frac{D\ln A}{Dt} + \frac{a}{2(k-1)c_p}\frac{\partial S}{\partial \beta} + F_2(a, V, x, t), \tag{2}$$

where $$r = \frac{a}{k-1} - \frac{V}{2}, \quad s = \frac{a}{k-1} + \frac{V}{2}$$

are the Riemann invariants, V is the velocity of flow, a the velocity of sound, A the cross-section area of the duct, S the specific entropy, α and β the characteristic directions, c_p the specific heat at constant pressure, and F_1, F_2 the coefficients representing the influence of the boundary layer.

2. A centred rarefaction wave in a duct of slowly varying cross-section

Neglecting boundary-layer influence, assuming that

$$A = A(x,t) = A^0 + \epsilon_A(x,t),$$

where $\epsilon_A \ll A^0$, and representing all parameters as the sum of parameters corresponding to a simple wave and disturbances caused by area change; we write $V = V^0 + \epsilon_V$ where $\epsilon_V \ll V^0$, $a = a^0 + \epsilon_a$ where $\epsilon_a \ll a^0$, $r = r^0 + \epsilon_r$ where $\epsilon_r \ll r^0$, and

$s = s^0 + \epsilon_s$ where $\epsilon_s \ll s^0$. After substituting these relations into (1) and (2) we get

$$\frac{\partial \epsilon_r}{\partial \alpha} \equiv \frac{\partial \epsilon_r}{\partial t} + (V^0 + a^0)\frac{\partial \epsilon_r}{\partial t} = -\frac{V^0 a^0}{2}\frac{D \ln \epsilon_A}{Dt} \equiv -\frac{V^0 a^0}{2}\psi(x,t), \tag{3}$$

$$\frac{\partial \epsilon_s}{\partial \beta} \equiv \frac{\partial \epsilon_s}{\partial t} + (V^0 - a^0)\frac{\partial \epsilon_s}{\partial t} = -\frac{V^0 a^0}{2}\psi(x,t), \tag{4}$$

where $\psi(x,t)$ is a given function of x and t.

These equations are independent linear differential equations of the first order. The coefficients of these equations can be calculated (see Courant & Friedrichs 1948) by the use of the simple wave relations (see figure 1). We get

$$V^0 = \frac{2}{k+1}\left[\frac{x}{t} - (k-1)s^0\right], \tag{5}$$

$$a^0 = \frac{k-1}{k+1}\left(\frac{x}{t} + 2s^0\right). \tag{6}$$

We thus obtain

$$\frac{\partial \epsilon_r}{\partial t} + \frac{x}{t}\frac{\partial \epsilon_r}{\partial x} = -\frac{k-1}{k+1}\left[\left(\frac{x}{t}\right)^2 - (k-3)\frac{x}{t}s^0 - 2(k-1)s^{0^2}\right]\psi(x,t), \tag{7}$$

$$\frac{\partial \epsilon_s}{\partial t} + \left[\frac{3-k}{k+1}\frac{x}{t} - \frac{4(k-1)}{k+1}s^0\right]\frac{\partial \epsilon_s}{\partial x} = -\frac{k-1}{k+1}\left[\left(\frac{x}{t}\right)^2 - (k-3)\frac{x}{t}s^0 - 2(k-1)s^{0^2}\right]\psi(x,t). \tag{8}$$

Integrating the first equation, we find the characteristics are

$$\frac{x}{t} = c = \text{const.}, \tag{9}$$

and along these characteristics

$$\epsilon_r = \frac{k-1}{k+1}\left[\frac{x}{t} - (k-3)s^0 - \frac{2t}{x}(k-1)s^{0^2}\right]\Psi + C_\alpha\left(\frac{x}{t}\right), \tag{10}$$

where
$$\Psi \equiv \Psi(x,t) = \int \psi\left(x, \frac{x}{c}\right)dx,$$

and the C_α are constants which take a different value on each characteristic α. This value we can calculate from the condition that for $x = L+0$, $\epsilon_r = 0$.

Assuming for simplicity that $\Psi \equiv \Psi(x)$, we get

$$\epsilon_r = -\frac{k-1}{k+1}\left[\frac{x}{t} - (k-3)s^0 - \frac{2t}{x}(k-1)s^{0^2}\right]\Psi(x) + \frac{k-1}{k+1}\left[\frac{L}{t} - (k-3)s^0 - \frac{2t}{L}(k-1)s^{0^2}\right]\Psi(L+0). \tag{11}$$

If, for the sake of example, $A/A_0 = e^{Kx}$, where K is constant, then

$$\epsilon_r = -\frac{k-1}{k+1}\left[\frac{x-L}{t} + \frac{2t(x-L)}{xL}(k-1)s^{0^2}\right]K. \tag{12}$$

A solution in closed form for ϵ_r could also have been obtained for a non-centred wave. Integrating equation (8), we find the β-characteristics of the centred wave are given by

$$x = C_1 t^{(3-k)/(k+1)} - 2s^0 t, \tag{13}$$

where $C_1 = \text{const.}$ along each β-characteristic.

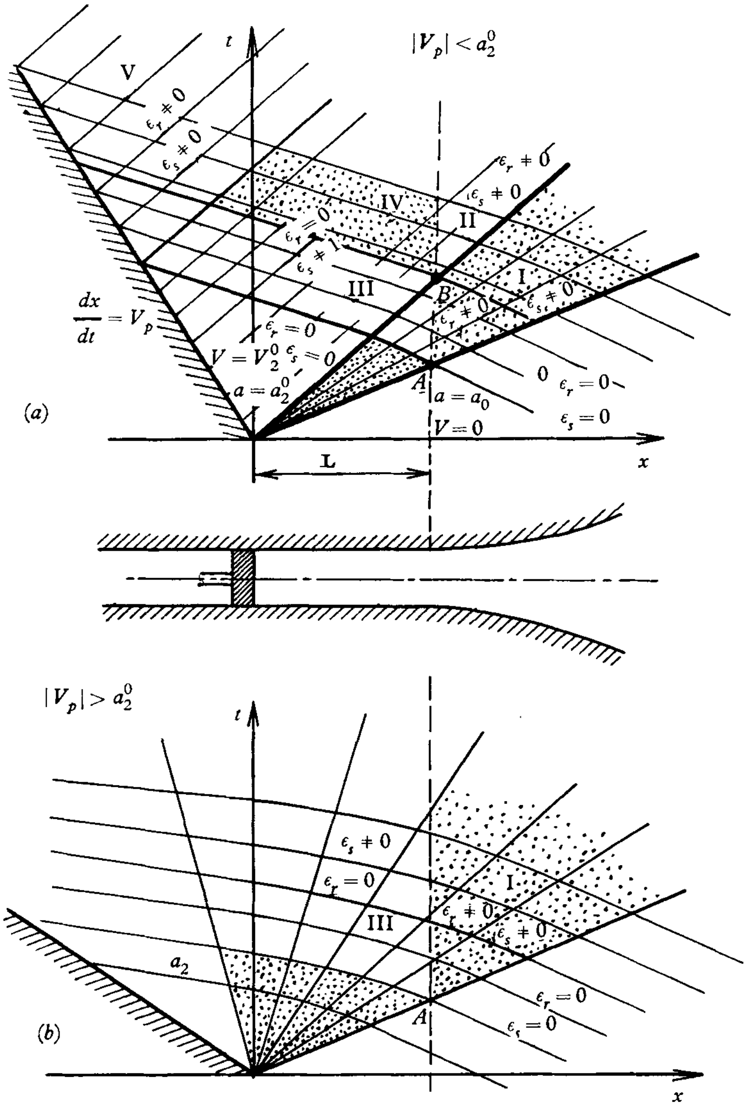

FIGURE 1. Propagation of a centred rarefaction wave through a duct of slowly variable cross-section. (*a*) Subsonic flow behind the rarefaction wave. (*b*) Supersonic flow behind the rarefaction wave.

It is of interest that we get a wave in which the characteristics as given by equations (9) and (13) are the same as for a simple wave (in a duct of constant cross-section). To the present approximation, the α and β characteristics coincide for waves in a duct with constant or slowly varying cross-section.

The change of ϵ_s along the β-characteristics is given by

$$\epsilon_s = -\frac{k-1}{k+1}\int \{[C_1 t^{[2(1-k)/(k+1)]} - 2s^0]^2 - (k-3)[C_1 t^{[2(k-1)/(k+1)]} - 2s^0]\, s^0 - 2(k-1)\, s^{0^2}\}\, \psi\, dt. \tag{14}$$

For simplicity we shall evaluate this integral only for a special duct with shape prescribed by

$$A = A^0 e^{Kx}, \tag{15}$$

where $K = \text{const.}$

We get, after substituting for C_1 from (13),

$$\epsilon_s = -\frac{k-1}{k+1}(x+2s^0t)\left[\frac{x+2s^0t}{(5-3k)t} - \frac{k+1}{3-k}s^0\right]K + C_\beta, \tag{16}$$

where C_β is constant along the β-characteristic and can be calculated from the boundary conditions on the front of the wave; namely, $V = 0$, $a = a_0$, $\epsilon_s = 0$ for $x/t = a_0$. Then we obtain

$$C_\beta = \frac{2Ka_0^2(k+1)t}{(5-3k)(3-k)}. \tag{17}$$

Equations (12) and (16) give the values of the deviation of the flow parameters from the simple wave solution in region I prescribed by $x \geqslant L$ (see figure 1) and

$$\frac{x}{a_0 + \frac{1}{2}(k+1)V_0^2} \geqslant t \geqslant \frac{x}{a_0},$$

where V_2^0 is the velocity on the back of the centred wave. In the special case when $|V_2^0| > a_2^0$, point B (figure 1*b*) is at infinity.

For the case of subsonic flow behind the centred wave, there is a reflected wave-region III (see figure 1*a*). In this region, $\epsilon_r = 0$ but $\epsilon_s \neq 0$. Along the β-characteristics, $\epsilon_s = C_\beta = \text{const.}$ The values of C_β are the same as at the points of intersection of the β-characteristics with the vertical line $x = L$. These values are given by equations (16) and (17)

$$C_{\beta AB} = -\frac{k-1}{k+1}(L+2s^0t_{AB})\left[\frac{L+2s^0t_{AB}}{(5-3k)t_{AB}} - \frac{k+1}{3-k}s^0\right]K + \frac{2a_0^2K(k+1)t_{AB}}{(5-3k)(3-k)}. \tag{18}$$

For an arbitrary point (L, t_{AB}), by the use of equation (13), the equation of the characteristic at the back of the centred wave, i.e. $x/t = V_2^0 + a_2^0$, and equation (25), we can calculate for given t the corresponding co-ordinate x lying on the β-characteristic which intersects the vertical line $x = L$ at the point (L, t_{AB}):

$$\begin{aligned} x - L = {} & (L+2s^0t_{AB})\left[\left(\frac{k+1}{k-1}\frac{t_{AB}}{L+2s^0t_{AB}}a_2^0\right)^{(3-k)/[2(1-k)]} - 1\right] \\ & + (V_2^0 - a_2^0)t + 2s^0t_{AB} - \frac{k+1}{k-1}a_2^0\left[\frac{t_{AB}^{(3-k)/(k+1)}}{L+2s^0t_{AB}}\frac{k+1}{k-1}a_2^0\right]^{(k+1)/[2(1-k)]} \end{aligned} \tag{19}$$

For the case of subsonic flow behind a centred wave ($|V_2^0| < a_2^0$), we can calculate the flow parameters in region II (figure 1*a*) from the system of equations similar to (3) and (4) but with constant coefficients

$$\frac{\partial \epsilon_r}{\partial \alpha} \equiv \frac{\partial \epsilon_r}{\partial t} + (V_2^0 + a_2^0)\frac{\partial \epsilon_r}{\partial x} = -\frac{V_2^0 a_2^0}{2}\psi, \tag{20}$$

$$\frac{\partial \epsilon_s}{\partial \beta} \equiv \frac{\partial \epsilon_s}{\partial t} + (V_2^0 - a_2^0)\frac{\partial \epsilon_s}{\partial x} = -\frac{V_2^0 a_2^0}{2}\psi. \tag{21}$$

Then we find from equation (20) that the α-characteristics are straight lines

$$x = (V_2^0 + a_2^0)\,t + x_{0\alpha}, \tag{22}$$

where $x_{0\alpha}$ is the co-ordinate of the point of intersection of the α-characteristic with the x-axis. Along these characteristics

$$\epsilon_r = -\frac{V_2^0 a_2^0}{2} K \frac{x}{V_2^0 + a_2^0} + C_\alpha, \tag{23}$$

where C_α is a constant along each characteristic which can be calculated from the boundary condition $\epsilon_r = 0$ when $x = L$. Then we get

$$\epsilon_r = -\frac{V_2^0 a_2^0}{2} K \frac{x-L}{V_2^0 + a_2^0}. \tag{24}$$

The characteristics of equation (21) can be written in the form

$$x = (V_2^0 - a_2^0)\,t + x_{0\beta}, \tag{25}$$

where $x_{0\beta}$ is the co-ordinate of the point of intersection of the β-characteristic with the x-axis. Along these characteristics, we have

$$\epsilon_s = -\frac{V_2^0 a_2^0}{2} Kt + C_\beta. \tag{26}$$

The constants C_β are calculated from the compatibility conditions for the solutions (16) and (26) along the rear boundary of the centred wave in regions I and II. We then have for $\quad x_b = (V_2^0 + a_2^0)\,t_b,$
the relation

$$-\frac{V_2^0 a_2^0}{2} Kt_b + C_\beta = -a_2^0 t_b \left[\frac{k+1}{k-1}\frac{a_2^0}{5-3k} - \frac{k+1}{3-k} s^0\right] K + \frac{2a_0^2 K(k-1)}{(5-3k)(3-k)} t_b.$$

The co-ordinates (x, t) of an arbitrary point lying in region II on the β-characteristic through the point (x_b, t_b) on the back of the centred wave satisfy

$$\frac{x_b - x}{t_b - t} = V_2^0 - a_2^0,$$

so that

$$t_b = \frac{x - (V_2^0 - a_2^0)\,t}{2a_2^0}.$$

Then finally we have

$$\epsilon_s = -\frac{V_2^0 a_2^0}{3-k} Kt + \frac{K(x - V_2^0 t)}{(3-k)\,a_0}\left[\frac{2(k+1)\,(a_0^2 - a_2^{0^2})}{5-3k} + (k-1)\,V_2^0 a_2^0\right]. \tag{27}$$

Equations (24) and (27) then give the flow parameters in region II (figure 1*a*).

In region IV (figures 1*a*), $\epsilon_r = 0$, and ϵ_s takes a constant value along β-characteristics which is equal to the value at the points of intersection of the β-characteristic with $x = L$. Then in region IV, we get

$$\epsilon_s = -\frac{V_2^0 a_2^0}{3-k} K\left(t - \frac{x-L}{V_2^0 - a_2^0}\right)$$
$$+ \frac{K\left(L - V_2^0 t + \dfrac{x-L}{V_2^0 - a_2^0} V_2^0\right)}{(3-k)\,a_2^0}\left[\frac{2(k+1)\,(a_0^2 - a^{0^2})}{5-3k} + (k-1)\,V_2^0 a_2^0\right]. \tag{28}$$

When the β-characteristics reach the moving piston, the boundary condition along the piston path, $x/t = V_2^0 = \text{const}$, gives $\epsilon_V = \epsilon_r - \epsilon_s = 0$. Then we have a reflected wave (region V on figure 1*a*) in which the α-characteristics are given by $dx/dt = V_2^0 + a_2^0$, and along which $\epsilon_r = \text{const}$. The value of ϵ_r is equal to the value of ϵ_s for the β-characteristic passing through the point where the α-characteristic meets the piston path.

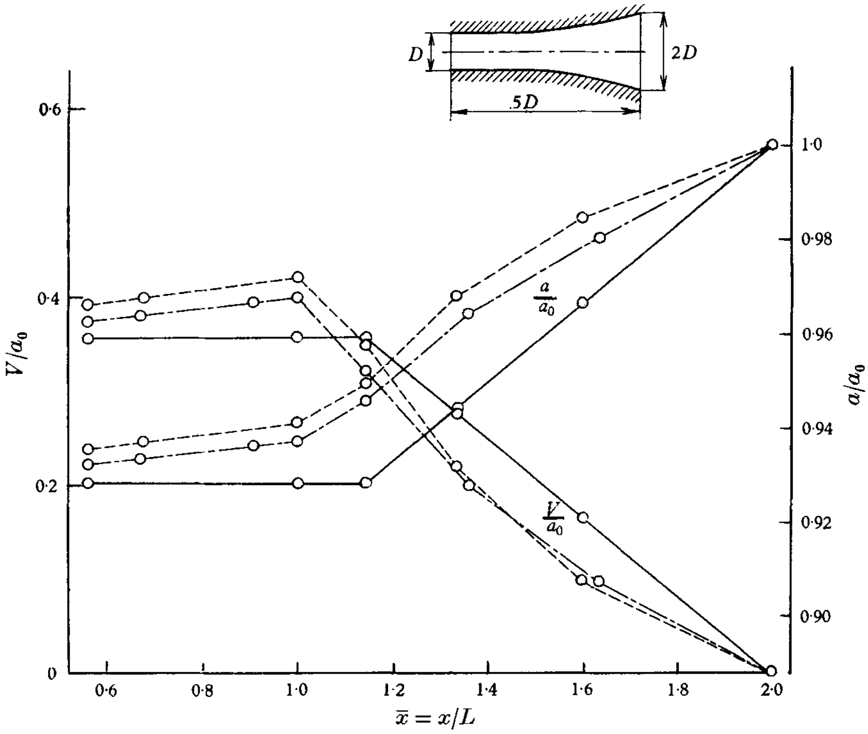

FIGURE 2. Comparison of the present method with a step-by-step calculation of the velocity and velocity of sound in a centred wave along a tube of variable cross-section given by $A/A_0 = e^{1\cdot 384(x/L)}$, when $a_0 t/L = 2$. ----, Step-by-step calculations; —·—, linearized theory; ——, simple wave.

A comparison of the present method with step-by-step calculations† using equations (1) and (2) is shown in figures 2 and 3 for $K = 1\cdot 384$ and $k = 1\cdot 4$, and for $tL/a_0 = 2$ and $tL/a_0 = 2\cdot 5$. The solid lines correspond to the simple wave solution for $A = A^0 = \text{const}$.

We thus find reasonably good results for a duct of non-slowly varying cross-section, in which K has a value such that when $\bar{x} = 5$, the cross-section area of the duct increases four times ($A/A_0 = 4$).

3. The boundary-layer influence on the simple wave

This problem was discussed by Owczarek (1956), who supposed that the boundary layer caused only surface friction and a production of entropy. In fact, the boundary-layer influence is more complicated, because of the change in

† These computations were made by P. Kijkowski.

the effective duct cross-section area. A more detailed consideration, based on the three principal equations of mechanics, of the boundary-layer influence in the special case of shock-tube flow was given by Rościszewski (1959). Below, we

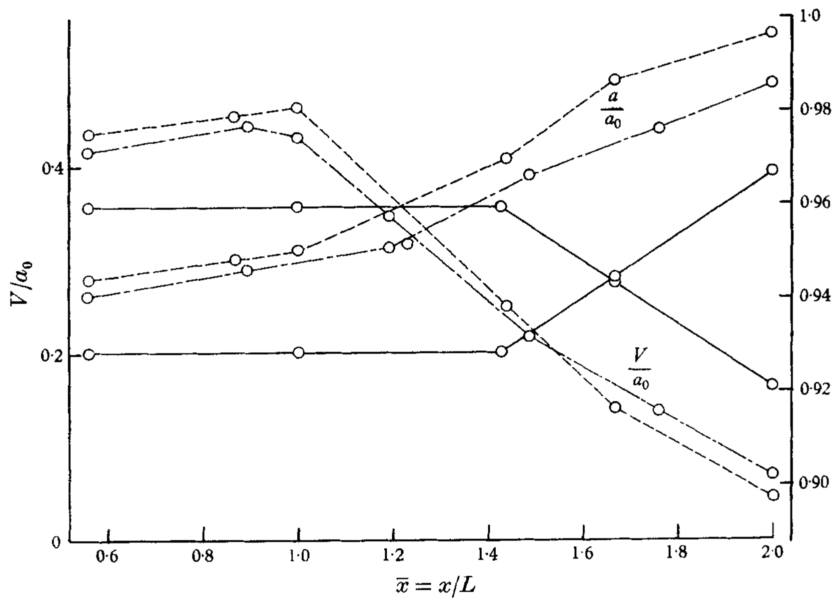

FIGURE 3. Comparison of the present method with a step-by-step calculation of the velocity and velocity of sound in a centred wave along a tube of variable cross-section given by $A/A_0 = e^{1\cdot 384(x/L)}$, when $a_0 t/L = 2\cdot 5$. ----, Step-by-step calculations; —·—, linearized theory; ——, simple wave.

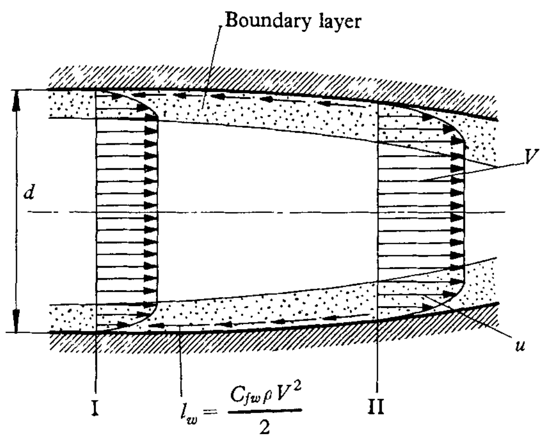

FIGURE 4. Boundary-layer influence on unsteady flow in a duct of slowly varying cross-section.

shall generalize this theory for the arbitrary one-dimensional unsteady flow. We shall assume that the boundary-layer thickness is very small in comparison with the duct diameter, and that the pressure is constant in each cross-section. We then apply the three basic equations of mechanics to the control volume between sections I and II of the tube (figure 4) of variable cross-section. We assume the variation of duct cross-section to be a small quantity of the same order as the variation of boundary-layer thickness.

The conservation of mass gives

$$\left[\iint_A \rho u dA\right]_{\mathrm{I}} - \left[\iint_A \rho u dA\right]_{\mathrm{II}} = \frac{\partial}{\partial t}\left\{\int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}}\left[\iint_A \rho dA\right] dx\right\},$$

where u and ρ are the velocity and density in a given cross-section. We note that

$$\iint_A \rho^i u^j dA = (A - \pi\delta_{ij} d)\bar{\rho}^i V^j,$$

where V and $\bar{\rho}$ are constant in a given cross-section and are the velocity and density outside of boundary layer, and

$$\delta_{ij} = \int_0^\delta \left(1 - \frac{\rho^i u^j}{\bar{\rho}^i V^j}\right) dy. \tag{29}$$

In particular $\delta_{11} \equiv \delta^*$ is the well-known boundary-layer displacement thickness.

Finally we get† for $x_{\mathrm{I}} \to x_{\mathrm{II}}$,

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\frac{\partial V}{\partial x} = -\bar{\rho}\frac{D \ln A}{Dt} + \frac{4\bar{\rho}}{d}\left(V\frac{\partial \delta^*}{\partial x} + \frac{\partial \delta_{10}}{\partial t}\right). \tag{30}$$

The momentum equation gives

$$(Ap)_{\mathrm{I}} - (Ap)_{\mathrm{II}} + \int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}} p\frac{\partial A}{\partial x} dx - \int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}} \pi d \frac{\bar{\rho} V^2}{2} C_{fw} dx$$
$$= \left[\iint_A \rho u^2 dA\right]_{\mathrm{II}} - \left[\iint_A \rho u^2 dA\right]_{\mathrm{I}} + \frac{\partial}{\partial t}\left\{\int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}}\left[\iint_A \rho u dA\right] dx\right\},$$

where C_{fw} is the surface friction coefficient $\left(\tau = \frac{\bar{\rho} V^2}{2} C_{fw}\right)$. Using equation (29) we get for $x_{\mathrm{I}} \to x_{\mathrm{II}}$,

$$\frac{DV}{Dt} + \frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} = -\frac{2C_{fw} V^2}{d} - \frac{4V^2}{d}\frac{\partial(\delta^* - \delta_{12})}{\partial x} - \frac{4V}{d}\frac{\partial(\delta_{10} - \delta^*)}{\partial t}. \tag{31}$$

The energy equation gives

$$\left[\iint_A i_0 \rho u dA\right]_{\mathrm{II}} - \left[\iint_A i_0 \rho u dA\right]_{\mathrm{I}} + \frac{\partial}{\partial t}\left\{\int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}}\left[\iint_A \rho E dA\right] dx\right\} = -\pi d \int_{x_{\mathrm{I}}}^{x_{\mathrm{II}}} q_w dx,$$

where
$$i_0 = \frac{u^2}{2} + \frac{k}{k-1}\frac{p}{\rho}$$
is the stagnation enthalpy,
$$E = \frac{u^2}{2} + \frac{1}{k-1}\frac{p}{\rho} \quad \text{and} \quad q_w$$

is the heat flow through the duct wall per unit time per unit area. Using equations (29) and (31), and
$$\frac{S - S_0}{C_V} = \ln\frac{p}{\rho^k},$$

† Assuming $A = \frac{1}{4}\pi d^2$.

we get for $x_{II} \to x_I$,

$$\frac{1}{C_p}\frac{DS}{Dt} = \frac{2(k-1)V^3}{da^2}\left[\frac{2}{k-1}\frac{a^2}{V^2}\frac{\partial \delta_{01}}{\partial x} + \frac{1}{V}\left(1-\frac{2}{k-1}\frac{a^2}{V^2}\right)\frac{\partial \delta_{10}}{\partial t} + \left(1-\frac{2}{k-1}\frac{a^2}{V^2}\right)\frac{\partial \delta^*}{\partial x} - \frac{2}{V}\frac{\partial \delta^*}{\partial t} - 2\frac{\partial \delta_{12}}{\partial x} + \frac{1}{V}\frac{\partial \delta_{12}}{\partial t} + \frac{\partial \delta_{13}}{\partial x} + C_{fw}\right] - \frac{4q_w(k-1)}{\rho a^2 d}. \quad (32)$$

Using equations (29), (31) and (32), we can express the coefficients F_1, F_2 in equations (1) and (2) as

$$F_{1,2} = \frac{2V^2}{d}\left\{\left(\pm 1 - \frac{k-1}{2}\frac{V}{a}\right)\frac{\partial \delta^*}{\partial x} + \left(\mp 1 - \frac{k-1}{a}V\right)\frac{\partial \delta_{12}}{\partial x} - \frac{a}{V}\frac{\partial \delta_{01}}{\partial x} - \frac{k-1}{2}\frac{V}{a}\frac{\partial \delta_{13}}{\partial x} - \left[\pm\frac{1}{V} - (k-1)\frac{1}{a}\right]\frac{\partial \delta^*}{\partial t} - \frac{1}{V}\left(\frac{k-1}{2}\frac{V}{a} \mp 1\right)\frac{\partial \delta_{10}}{\partial t} - \frac{k-1}{2}\frac{1}{a}\frac{\partial \delta_{12}}{\partial x} \pm \frac{c_{fw}}{2}\left[1 \mp (k-1)\frac{V}{a}\right] + \frac{(k-1)q_w}{\rho a V^2}\right\}. \quad (33)$$

We assume that the boundary-layer parameters δ_{ij}, c_{fw} and q_w are given on the basis of the solution of the boundary-layer equations with the velocity in the duct given by the ideal flow solution. This is a good approximation when the boundary layer is thin. For the case when it is not so, we can apply the method of successive approximation.

In particular, by making use of the theory of the non-steady boundary layer caused by a simple wave and of equation (33), it is possible to compute (as before) the boundary-layer influence on the centred wave. For example, it is possible to employ Backer's (1957) results which are obtained by the assumption of constant pressure, density and temperature in a given cross-section. In this case, using a very close approximation to Backer's formula (43) ($g = 4\eta + 2\eta^2$) for laminar flow and Rayleigh's approximation for turbulent flow, we obtain the solution in closed form for the centred wave. All the boundary-layer characteristics in this case depend in a simple manner of $\eta = 1 - x/a_0 t$ and t.

The integrals on the right-hand side of the equations obtained, which are similar to equation (14), can be solved in closed form. Backer's results can be used only for the weak waves (a small velocity of flow) because of the assumption of constant density in a cross-section.

For a more general solution of the problem discussed it is necessary to solve the unsteady compressible boundary layer, with the assumption of a simple wave potential flow and a heat conducting duct wall.

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